



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Discrete Mathematics 266 (2003) 119–131

DISCRETE
MATHEMATICSwww.elsevier.com/locate/disc

An improved upper bound for queens domination numbers

A.P. Burger, C.M. Mynhardt*

*Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045,
Victoria, BC Canada V8W 3P4*

Received 2 April 2001; received in revised form 13 September 2001; accepted 12 August 2002

Abstract

We consider the domination number of the queens graph Q_n and show that if, for some fixed k , there is a dominating set of Q_{4k+1} of a certain type with cardinality $2k+1$, then for any n large enough, $\gamma(Q_n) \leq [(3k+5)/(6k+3)]n + \mathcal{O}(1)$. The same construction shows that for any $m \geq 1$ and $n = 2(6m-1)(2k+1) - 1$, $\gamma(Q_n^t) \leq [(2k+3)/(4k+2)]n + \mathcal{O}(1)$, where Q_n^t is the toroidal $n \times n$ queens graph.

© 2003 Elsevier Science B.V. All rights reserved.

MSC: 05C69

Keywords: Chessboard; Toroidal chessboard; Queens graph; Queens domination problem

1. Introduction

We generally follow the notation and terminology pertaining to domination of [11]. We repeat the main concepts for chessboards here. The *queens graph* Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are *adjacent* if they are in the same *line*, i.e., row, column or diagonal. A queen on square x of Q_n *covers* a square y if x and y are adjacent. A set D of squares is a *dominating set* of Q_n if every square of Q_n is either in D or adjacent to a square in D , i.e., if a set of queens, one on each square in D , covers the board. If no two squares of the dominating set D are adjacent, then D is an *independent dominating set*. The *domination number* $\gamma(Q_n)$ (*independent domination number* $i(Q_n)$) of Q_n is the minimum size amongst all

* Corresponding author. Fax: +1-250-721-8962.

E-mail addresses: alewyn@math.uvic.ca (A.P. Burger), mynhardt@math.uvic.ca (C.M. Mynhardt).

dominating (independent dominating) sets of Q_n . It is easily seen that $\gamma(Q_n) \leq i(Q_n)$ for all n .

The problem of determining $\gamma(Q_n)$ and $i(Q_n)$ was first posed in 1862 [9] and, after little progress for over a century, has received considerable attention over the last 15 years. The first breakthrough was by P.H. Spencer, as cited in [8], who proved the lower bound $\gamma(Q_n) \geq \frac{1}{2}(n-1)$, $n \geq 1$. Refining Spencer's method, Weakley [15] improved this bound for $n \equiv 1 \pmod{4}$ by proving that $\gamma(Q_{4k+1}) \geq 2k+1$ for all $k \geq 0$. He also showed [17] that if $n < 143$, then $\gamma(Q_n) \geq n/2$. These bounds have enabled researchers [1,2,4,5,10,12,15,17] and especially [14] to determine exact values of $\gamma(Q_n)$ and $i(Q_n)$ to the extent that for $n \leq 120$, each of $\gamma(Q_n)$ and $i(Q_n)$ is either known, or known to have one of two values.

Thus we have ample evidence that at least for small values of n , $\gamma(Q_n) \approx n/2$. General upper bounds that are close to $n/2$ seem to be harder to find. Recent upper bounds were given in [6,14,16]. In particular it was shown (amongst other bounds) in [16] that if, for some fixed k , there is a dominating set of cardinality $2k+1$ of Q_{4k+1} with certain properties, then for n large enough, $\gamma(Q_n) \leq [(2k+4)/(4k+3)]n + \mathcal{O}(1)$.

In this paper we show that if there is such a dominating set of Q_{4k+1} with cardinality $2k+1$, then $\gamma(Q_n) \leq [(3k+5)/(6k+3)]n + \mathcal{O}(1)$. Thus as dominating sets of this type are found for increasing values of k , the general upper bound improves. Since $[(3k+5)/(6k+3)] < [(2k+4)/(4k+3)]$ for all $k \geq 4$, this bound is an improvement of the previous asymptotic bound for dominating sets of this type.

2. Dominating sets of Q_{4k+1}

Since we only consider odd integers n , we identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, with sides parallel to the coordinate axes and centre at the origin. We refer to the squares of the board by the coordinates of their centres; thus square (x, y) is in column x and row y . Rows and columns are collectively called *orthogonals*. An orthogonal is called *even* or *odd* according to the parity of its label, while a square is called *even-even*, *odd-odd*, *even-odd* or *odd-even* according to the parity of its coordinates. We consider placements of queens on even orthogonals only, with at least one queen in each even orthogonal, called *Type E sets* of queens. Thus all the squares in these orthogonals are dominated and the only squares that need to be considered are the odd-odd squares, which must be dominated diagonally.

We simplify the representation by drawing only the odd-odd squares, while the even rows and even columns can be considered to be squeezed to form lines. (See [6, Fig. 2].) Thus the coordinates $(2x, 2y)$ of an even-even square on the board correspond to the coordinates (x, y) of the intersection of its orthogonals on the reduced board. Henceforth, when we refer to coordinates, it will be of the simplified representation. As in the case of squares of the chessboard, a queen in the simplified representation is called *even-even*, *even-odd*, *odd-even* or *odd-odd* according to the parity of her coordinates.

The diagonals (of squares) that rise from left to right correspond to the straight lines with equations $y = x + d$, where $d \in \{-2k, \dots, -1, 0, 1, \dots, 2k\}$. These lines are called *d-diagonals* and are labelled $d = -2k, \dots, d = -1, d = 0, d = 1, \dots, d = 2k$ according to their intersections with the y -axis. Similarly, the *s-diagonals* fall from left to right, and correspond to the straight lines with equations $y = -x + s$, $s \in \{-2k, \dots, -1, 0, 1, \dots, 2k\}$ and are also labelled according to their intersections with the y -axis. An *even (odd) diagonal* is a diagonal with an even (odd) intersection with the y -axis. Notice that a queen (in an even row and column of the unreduced board) that lies on an odd (even) *d*-diagonal, also lies on an odd (even) *s*-diagonal and *vice versa*. Queens on odd (even) diagonals are also referred to as *odd (even) queens*. By a *point on a d-diagonal (or s-diagonal)* we mean an intersection point of its corresponding line and a line formed by an even orthogonal. Notice that the difference between the y and x coordinates of any point on a *d*-diagonal is equal to its label. Similarly, the sum of the coordinates of any point on the *s*-diagonals is equal to its label.

A line (row, column, diagonal, orthogonal) which does not contain a queen is called an *empty line (row, column, diagonal, orthogonal)*. A Type **E** set of (at least $2k + 1$) queens on Q_{4k+1} , where the first empty *s*-diagonal from the centre is $s = i$ or $s = -i$ and the first empty *d*-diagonal from the centre is $d = j$ or $d = -j$, is called an (i, j) -set, $i, j \geq 0$. As shown in [1], the only dominating (i, j) -sets of Q_{4k+1} with cardinality $2k + 1$ are (up to isomorphism) (i, i) - and $(i, i + 2)$ -sets. We need the following results from [1].

Proposition 1 (Burger et al. [1]). *If D is a Type **E** dominating set of Q_{4k+1} and there are no queens on the diagonal $d = i$ (respectively $s = i$), then there are queens on*

$$s \text{ (respectively } d) = 0, \pm 2, \pm 4, \dots, \pm(i - 1), \pm(i + 1), \dots,$$

$$\pm(2k - |i| - 1), \quad i \text{ odd}$$

$$s \text{ (respectively } d) = \pm 1, \pm 3, \dots, \pm(i - 1), \pm(i + 1), \dots,$$

$$\pm(2k - |i| - 1), \quad i \text{ even.}$$

Using Proposition 1, one obtains the following result.

Theorem 2 (Burger et al. [1]). (a) *An (i, i) -set of queens on Q_{4k+1} which lie on the diagonals*

$$s, d = 0, \pm 1, \pm 2, \dots, \pm(i - 1), \pm(i + 1), \pm(i + 3), \dots, \pm(2k - i - 1) \quad (1)$$

is dominating. Conversely, if D with $|D| = 2k + 1$ is an (i, i) -dominating set of Q_{4k+1} , then D contains queens on each of the diagonals in (1).

(b) *An $(i, i + 2)$ -set of queens on Q_{4k+1} which lie on the diagonals*

$$s = 0, \pm 1, \pm 2, \dots, \pm(i - 1), \pm(i + 1), \pm(i + 3), \dots, \pm(2k - i - 3), \quad (2)$$

$$d = 0, \pm 1, \pm 2, \dots, \pm i, \pm(i + 1), \pm(i + 3), \dots, \pm(2k - i - 1), \quad (3)$$

is dominating. Conversely, if D with $|D| = 2k + 1$ is an $(i, i + 2)$ -dominating set of Q_{4k+1} , then D contains queens on each of the diagonals in (2) and (3).

By again using Proposition 1 we obtain the following generalisation of Theorem 2(b). The proof is straightforward (similar to the proof of the relevant part of Theorem 2) and is omitted.

Proposition 3. For any $t \in \{1, 2, \dots, k - i\}$, an $(i, i + 2t)$ -set of (at least $2k + 2t - 1$) queens on Q_{4k+1} which lie on the diagonals

$$\begin{aligned} s &= 0, \pm 1, \pm 2, \dots, \pm(i - 1), \pm(i + 1), \pm(i + 3), \dots, \pm(2k - i - 2t - 1), \\ d &= 0, \pm 1, \pm 2, \dots, \pm(i + 2t - 1), \pm(i + 2t + 1), \pm(i + 2t + 3), \dots, \pm(2k - i - 1), \end{aligned}$$

is dominating.

Depending on the parity of i , there are either more odd queens or more even queens in a dominating (i, j) -set D of Q_{4k+1} . Call the smaller of these sets the *core* of D and the bigger one the *body* of D . The *core diagonals* (respectively *body diagonals*) are the diagonals listed in (1) containing core (respectively body) queens. Note that there can be core (body) queens that are not on the core (body) diagonals, i.e., they lie on diagonals not listed in (1).

3. Construction and lemmas

We now describe the construction we use to create dominating sets for the new upper bound.

Construction. For some $k \geq 2$, let D be a dominating set of Q_{4k+1} of cardinality $2k + 1$. For any $m \geq 1$, let $k' = k(6m - 1) + (3m - 1)$, $n' = 4k' + 1 = 2(6m - 1)(2k + 1) - 1$ and $i' = (6m - 1)(i + 1)$. A *copy on queen* $(x, y) \in D$ is obtained by placing queens on the points in the plane with coordinates

$$(6m - 1)(x, y) \pm (j, 6m - 1 - 3j) \quad \text{for } j = 1, 2, \dots, 3m - 1. \quad (4)$$

Define D' as the union of the copies on all $(x, y) \in D$.

Thus D' consists of $2k + 1$ copies of $6m - 2$ queens each, some of which do not lie on $Q_{n'}$. We call the queens of D' on (not on, respectively) $Q_{n'}$ *feasible (infeasible) queens*. The set of black dots in Fig. 1 is an example of D' with $m = 2$, obtained from the dominating $(2, 2)$ -set $D = \{(0, 0), \pm(1, 2), \pm(2, -1), \pm(3, 3)\}$ of Q_{13} ($k = 3$). For clarity only the even rows and columns of the reduced board are drawn for most of the board. Let the *centre* of a copy on $(x, y) \in D$ be the square with coordinates $(6m - 1)(x, y)$. We will show that by modifying D' slightly and by adding a number of queens we obtain a dominating set of $Q_{n'}$. We need the following lemmas.

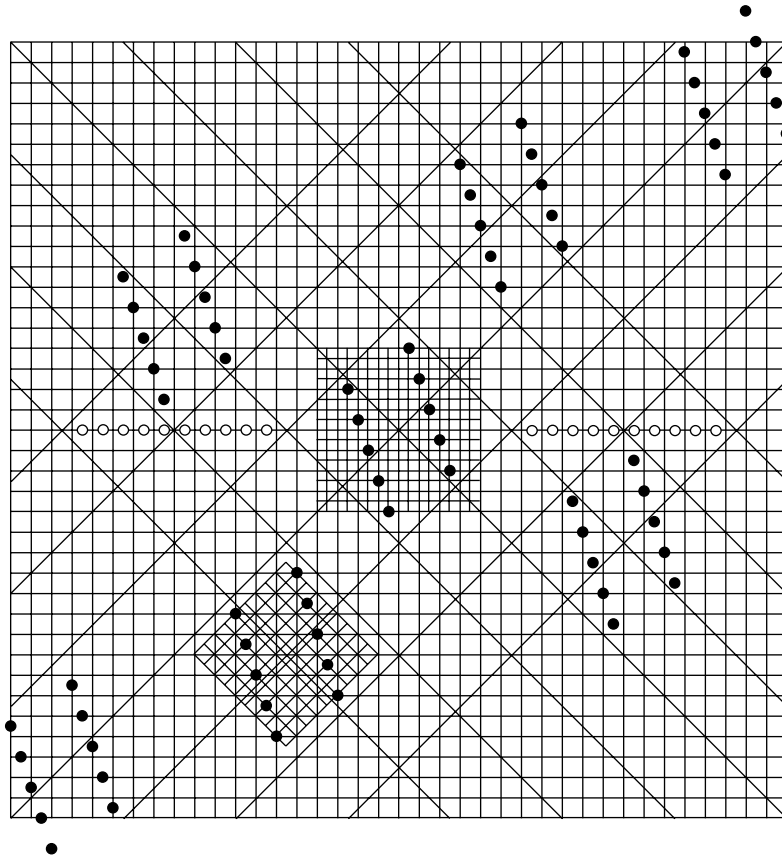


Fig. 1. An example of D' with $m = i = 2$ and $k = 3$.

Lemma 4. *Each copy on a queen occupies $6m - 2$ consecutive s and d diagonals of the same parity, these diagonals being symmetric around the center of each copy.*

Proof. Consider any queen $(x, y) \in D$. The sum diagonals of the copy on (x, y) are given by

$$(6m - 1)(y + x) \pm (6m - 1 - 2j) \quad \text{for } j = 1, 2, \dots, 3m - 1,$$

which are clearly consecutive and of the same parity. It is also clear that $3m - 1$ of the diagonals are above the sum diagonal going through the centre of the copy and $3m - 1$ below. Thus the sum diagonals are symmetric around the centre.

The difference diagonals are given by

$$(6m - 1)(y - x) \pm (6m - 1 - 4j) \quad \text{for } j = 1, 2, \dots, 3m - 1.$$

These diagonals are also symmetric around the centre. To see that they are consecutive we note that the numbers $j = 3m - 1, 1, 3m - 2, 2, 3m - 3, 3, \dots, \lfloor (3m - 1)/2 \rfloor$ correspond

to the diagonals

$$(6m-1)(y-x) \pm (6m-3)$$

$$(6m-1)(y-x) \pm (6m-5)$$

$$\vdots$$

$$(6m-1)(y-x) \pm 1$$

respectively, which are consecutive and of the same parity. \square

Lemma 5. *The copies on two queens of D on consecutive diagonals of the same parity, say s (respectively d) $=l$ and s (respectively d) $=l+2$, cover consecutive s - (respectively d -) diagonals of the same parity of Q_n , except for the diagonal s (respectively d) $= (6m-1)(l+1)$ between the copies.*

Proof. Consider two queens on the diagonals $d=l$ and $l+2$. Let L and $L+2$, respectively, be the corresponding copies in D' . By Lemma 4, each copy covers consecutive diagonals of the same parity. The largest label of the d -diagonals in L is $(6m-1)l+6m-3$, while the smallest label of the d -diagonals in $L+2$ is $(6m-1)(l+2)-(6m-3)=(6m-1)l+6m+1$. Thus the queens of the two copies cover consecutive diagonals except for the diagonal $d=(6m-1)(l+1)$. The proof for the s -diagonals is similar. \square

Lemma 6. *If two queens in D are on different lines of Q_{4k+1} , then all the queens of the corresponding copies in D' are on different lines (or their extensions) of Q_n .*

Proof. We first consider the diagonals. If two queens in D are on s -diagonals (d -diagonals, respectively) of the same parity p , then the corresponding queens in D' are also on s -diagonals (d -diagonals, respectively) of the same parity $1-p$. Thus if two queens have different parity in D then the corresponding queens in D' will be on different diagonals. If queens are of the same parity, then by Lemma 5 the diagonals of the corresponding copies are also different.

Consider queens $(x, y), (x', y') \in D$ with copies C, C' respectively, in D' . If $x'=x+1$, i.e., if the queens are on consecutive columns of Q_{4k+1} , then the largest column of a queen in C is $(6m-1)x+3m-1$, while the smallest column of a queen in C' is $(6m-1)(x+1)-(3m-1)=(6m-1)x+3m$. If $y'=y+1$, i.e., if the queens are on consecutive rows of Q_{4k+1} , then the rows of queens in $C \cup C'$ are labelled

$$(6m-1)y + (6m-1-3j) \quad \text{for } j=1, 2, \dots, 3m-1, \quad (5)$$

$$(6m-1)y - (6m-1-3j) \quad \text{for } j=1, 2, \dots, 3m-1, \quad (6)$$

$$(6m-1)(y+1) + (6m-1-3j) \quad \text{for } j=1, 2, \dots, 3m-1, \quad (7)$$

$$(6m-1)(y+1) - (6m-1-3j) \quad \text{for } j=1, 2, \dots, 3m-1. \quad (8)$$

Note that the labels in (5) (respectively (6), (7), (8)) are congruent to $-y + 2$ (respectively $-y + 1$, $-y + 1$, $-y$) modulo 3. Thus the only labels that can be the same are those in (6) and (7). The largest label in (6) and the smallest label in (7) occur when $j = 3m - 1$, giving $(6m - 1)y + 3m - 2$ and $(6m - 1)y + 3m + 1$, respectively. Thus no labels are the same. \square

The following lemma gives the number of infeasible queens.

Lemma 7. *There are $2(m - 1)$ infeasible queens in D' .*

Proof. The label of the last positive row and column on $Q_{n'}$ is

$$x = y = k' = k(6m - 1) + (3m - 1).$$

The largest column label of a queen in D' (obtained by taking $x = k$ and $j = 3m - 1$ in Eq. (4)) is $x = k'$. Thus the only queens not on the board are those for which $|y| > k'$. The largest row label of a queen in D is k ; the rows of the corresponding copy in D' are $k(6m - 1) \pm (6m - 1 - 3j)$ for $j = 1, 2, \dots, 3m - 1$. Thus for $|(6m - 1 - 3j)| > 3m - 1$, i.e., $j < m$ and $j > 3m - \frac{2}{3}$, the queens do not lie on $Q_{n'}$. By symmetry there are $2(m - 1)$ infeasible queens. \square

Lemma 8. *D' has queens on every second row and column, or its extension, of $Q_{n'}$ (every orthogonal of the reduced board) except on the orthogonals through the centre of each copy and the rows with (reduced) labels*

$$y = \pm(k' - 2), \pm(k' - 5), \dots, m - 1 \text{ terms.} \quad (9)$$

Proof. It is clear that there are no queens on the columns through the centre of each copy. Not counting these, there are $(2k' + 1) - (2k + 1) = (6m - 2)(2k + 1)$ (reduced) columns. But $|D'| = (2k + 1)(6m - 2)$ and thus by Lemma 6 there are queens on all columns (or their extensions) not through the centres. The case for the rows is similar except that there are $2(m - 1)$ infeasible queens. Thus there are $2(m - 1)$ rows without queens. These rows are the rows that would contain feasible queens of copies on squares with y -coordinates $y = \pm(k + 1)$, which by definition are the rows in (9). \square

Lemma 9. *At most $4(m - 1)$ additional queens on $Q_{n'}$ are required to cover all lines of $Q_{n'}$ covered by infeasible queens and the rows given in (9).*

Proof. Suppose the infeasible queens are on the copies on (x_θ, k) and $(x_\theta, -k)$. Their coordinates are

$$(6m - 1)(x_\theta, k) + (j, 6m - 1 - 3j) \quad \text{for } j = 1, 2, \dots, m - 1, \quad (10)$$

$$(6m - 1)(x_\theta, -k) - (j, 6m - 1 - 3j) \quad \text{for } j = 1, 2, \dots, m - 1. \quad (11)$$

Let X_θ , S_θ and D_θ be the sets of columns, s -diagonals and d -diagonals of the squares in (10), respectively; define X_θ , S_θ and D_θ similarly. Each line in X_θ (X_θ , respectively)

intersects a line in S_θ (S_θ) or a line in D_θ (D_θ). These intersecting lines can be covered by $2(m-1)$ queens. The remaining lines can be covered by placing queens on the intersections of these with the rows listed in (9). It is easy to see this is always possible except in the extreme case when we have a copy on (k, k) or $(-k, -k)$ (as in Fig. 1). The problem can be solved by reflecting the copy vertically through its centre; the copy thus obtained covers precisely the same lines as before, but the s -diagonals of the infeasible queens are closer to the centre of the board. This gives us a total number of $4(m-1)$ additional queens. \square

4. Main theorem

Theorem 10. *If there exists a dominating (i, i) - or $(i, i+2)$ -set ($i \leq k-2$) of Q_{4k+1} of cardinality $2k+1$, then $\gamma(Q_{n'}) \leq [(3k+5)/(6k+3)]n' + \mathcal{O}(1)$.*

Proof. Let D^* consist of the feasible queens in D' and the $4(m-1)$ queens of Lemma 9; note that $|D^*| = (2k+1)(6m-2) + 2(m-1)$. Suppose first that D is an (i, i) -set. Recall from the construction of D' in Section 3 that $i' = (6m-1)(i+1)$. We extend D^* to a Type E set of queens on $Q_{n'}$ with at least one queen on each of the diagonals

$$s, d = 0, \pm 1, \pm 2, \dots, \pm(i' - 1), \pm(i' + 1), \dots, \pm(2k' - i' - 1), \quad (12)$$

which by Theorem 2 is a dominating (i', i') -set. By Lemma 8 and the proof of Lemma 9 all orthogonals contain queens in D^* except those through the centres of the copies. We consider the core and body diagonals separately. Since all body diagonals of D up to $2k-i-1$ are covered (by Theorem 2(a)), the positive body diagonals of $Q_{n'}$ covered by D^* are all diagonals up to

$$\begin{aligned} s', d' &= (2k-i-1)(6m-1) + (6m-3) \\ &= (6m-1)(2k-i) - 2 \\ &= 2k' - i' - 1, \end{aligned}$$

except for the diagonals between copies (see Lemma 5). Similarly, the positive core diagonals are all covered up to

$$\begin{aligned} s', d' &= (i-2)(6m-1) + (6m-3) \\ &= (i+1)(6m-1) - 2(6m-1) - 2 \\ &= i' - 2(6m-1) - 2 \end{aligned}$$

except for the diagonals between copies. The negative diagonals are similar. Let

$$M = \{\pm(i' - 2j, 0) : j = 1, 2, \dots, 6m-2\};$$

note that since $i \leq k-2$, M is contained in $Q_{n'}$. By placing queens on the squares in M , we cover almost all the remaining core diagonals in (12), i.e., from $s', d' = \pm(i' - 2(6m-2))$ to $s', d' = \pm(i' - 2)$ (see white dots in Fig. 1). The only exceptions are $s', d' = \pm(i' - 2(6m-1)) = \pm(i-1)(6m-1)$, which are dealt with in (14).

By Lemma 8 the orthogonals

$$y, x = 0, \pm 1(6m - 1), \pm 2(6m - 1), \dots, \pm k(6m - 1) \quad (13)$$

through the centres of the copies must be covered. Also, all the diagonals between copies, where we regard M as a copy, must be covered. They are

$$\begin{aligned} s', d' = 0, \pm 1(6m - 1), \dots, \pm i(6m - 1), \pm (i + 2)(6m - 1), \dots, \\ \pm (2k - i - 2)(6m - 1). \end{aligned} \quad (14)$$

Notice that all the labels of the lines in (13) and (14) are multiples of $6m - 1$ and that there are $2k + 1$ ($2k - 1$, respectively) labels for each type of orthogonal (diagonal, respectively), i.e., the number of lines is independent of m . To cover these lines we need at least $2k + 1$ and at most $6k - 1$ queens; that is, a set R consisting of a constant number of queens. (For example, we may take an $(i + 1, i + 1)$ -dominating set of Q_{4k+1} , if it exists, and multiply each coordinate with $6m - 1$; otherwise, in the worst case we may place a queen on each of $2k + 1$ intersections of distinct orthogonals and one on each diagonal.)

It is clear that $X = D^* \cup M \cup R$ satisfies the hypothesis of Theorem 2 and hence is a dominating set of $Q_{n'}$ with

$$\begin{aligned} |X| &= |D^*| + |M| + |R| \\ &= (2k + 1)(6m - 2) + 2(m - 1) + 2(6m - 2) + \text{a constant} \\ &= 4m(3k + 5) + \text{a constant}. \end{aligned}$$

Now suppose that D is an $(i, i + 2)$ -set. We extend D^* to a Type E set of queens on $Q_{n'}$ with at least one queen on each of the diagonals

$$\begin{aligned} s = 0, \pm 1, \pm 2, \dots, \pm((6m - 1)(i + 1) - 1), \pm((6m - 1)(i + 1) + 1), \\ \pm((6m - 1)(i + 1) + 3), \dots, \pm(2k' - (6m - 1)(i + 3) - 1) \end{aligned} \quad (15)$$

$$\begin{aligned} d = 0, \pm 1, \pm 2, \dots, \pm((6m - 1)(i + 3) - 1), \pm((6m - 1)(i + 3) + 1), \\ \pm((6m - 1)(i + 3) + 3), \dots, \pm(2k' - (6m - 1)(i + 1) - 1). \end{aligned} \quad (16)$$

By Proposition 3 this will be a dominating $(i', i' + 2t)$ -set of $Q_{n'}$ with $i' = (6m - 1)(i + 1)$ and $t = 6m - 1$. We proceed as in the case where D is an (i, i) -set and note that all orthogonals contain queens in D^* except those through the centres of the copies. Using Theorem 2(b) and the construction, we see that the positive body diagonals of $Q_{n'}$ covered by D^* are all diagonals up to

$$\begin{aligned} s' &= (2k - i - 3)(6m - 1) + (6m - 3) = 2k' - (6m - 1)(i + 3) - 1 \\ &= 2k' - (i' + 2t) - 1, \\ d' &= (2k - i - 1)(6m - 1) + (6m - 3) = 2k' - (6m - 1)(i + 1) - 1 = 2k' - i' - 1, \end{aligned}$$

except for the diagonals between copies. Similarly, the positive core diagonals covered by D^* are all diagonals up to

$$s' = (i - 2)(6m - 1) + (6m - 3) = i' - 2(6m - 1) - 2 = i' - 2t - 2,$$

$$d' = i(6m - 1) + (6m - 3) = i' - 2,$$

except for the diagonals between copies. Thus to satisfy (15) and (16), we still need to cover the positive core diagonals

$$s' = i' - 2t, i' - 2t + 2, \dots, i' - 2, \quad (17)$$

$$d' = i', i' + 2, \dots, i' + 2t - 2. \quad (18)$$

A similar statement holds for the negative diagonals. Let

$$M' = \{\pm(1 - 2j, i' - 1) : j = 1, 2, \dots, t\}.$$

Since $k \geq 2$ and $i \leq k - 1$, M' is contained in $Q_{n'}$. By placing queens on the squares in M' , we cover all the core diagonals in (17) and (18). Proceeding as in the case where D is an (i, i) -set, we see that we can cover the orthogonals through the centres of the copies and the diagonals between the copies with a set R' consisting of a constant number of queens. Thus the set $X' = D^* \cup M' \cup R'$ satisfies the hypothesis of Proposition 3 and hence is a dominating set of $Q_{n'}$ with

$$\begin{aligned} |X'| &= |D^*| + |M'| + |R'| \\ &= (2k + 1)(6m - 2) + 2(m - 1) + 2(6m - 1) + \text{a constant} \\ &= 4m(3k + 5) + \text{a constant}. \end{aligned}$$

Since $n' = 2(6m - 1)(2k + 1) - 1 = 4m(6k + 3) + \text{a constant}$, we have shown in each case that $\gamma(Q_{n'}) \leq [(3k + 5)/(6k + 3)]n' + \mathcal{O}(1)$. \square

Note that there are restrictions on n' , but the set of admissible values of n' is an arithmetic progression, and for all values of n' we can create a dominating set by adding queens to a dominating set on a largest board of admissible size less than n' . At most one queen is needed for each new row and column. Therefore the number of queens added is never more than a constant, and we have

Corollary 11. *If there exists a dominating (i, i) - or $(i, i + 2)$ -set ($i < k$) of Q_{4k+1} of cardinality $2k + 1$, then for all n large enough, $\gamma(Q_n) \leq [(3k + 5)/(6k + 3)]n + \mathcal{O}(1)$.*

This improves the bound $\gamma(Q_n) \leq [(2k + 4)/(4k + 3)]n + \mathcal{O}(1)$ of [16] obtained from dominating (i, j) -sets of Q_{4k+1} . Weakley [16] also showed that if there is a dominating set of Q_{4k-1} of a certain type with cardinality $2k$, then $\gamma(Q_n) \leq [(2k + 3)/(4k + 1)]n +$

$\mathcal{O}(1)$. Since a dominating set of Q_{131} with the required properties exists (see [14]), we have $\gamma(Q_n) \leq \frac{69}{133}n + \mathcal{O}(1)$, the previously best bound.

The largest k for which an (i, i) -dominating set of Q_{4k+1} of cardinality $2k+1$ is known to exist, is $k=28$ ($i=12$) [14], which gives the bound $\gamma(Q_n) \leq \frac{89}{171}n + \mathcal{O}(1)$ when substituted in Corollary 11. On the other hand, the largest k for which an $(i, i+2)$ -dominating set of Q_{4k+1} of cardinality $2k+1$ is known to exist, is $k=32$ ($i=13$) [14]. This gives the following bound when substituted in Corollary 11, a slight improvement on $\gamma(Q_n) \leq \frac{69}{133}n + \mathcal{O}(1)$.

Corollary 12. *For all n large enough, $\gamma(Q_n) \leq \frac{101}{195}n + \mathcal{O}(1)$.*

5. Toroidal chessboards

Consider an $n \times n$ chessboard on the torus and notice that the rows and columns of the chessboard are rings on the torus. The *lines* of the board are the rows, columns, s -diagonals (i.e., sets of squares such that $x + y \equiv k \pmod{n}$, where k is a constant) and d -diagonals (sets of squares such that $y - x \equiv k \pmod{n}$). Note that there are n s -diagonals and n d -diagonals, and each contains n squares. The vertices of Q_n^t , the *queens graph obtained from an $n \times n$ chessboard on the torus*, are the n^2 squares of the chessboard, and two squares are adjacent if they are collinear. The queens domination problem for chessboards on the torus is addressed in [3,7,13], where it is shown that

$$\gamma(Q_{3k}^t) = \begin{cases} k & \text{if } k \equiv 1, 5, 7, 11 \pmod{12} \\ k+1 & \text{if } k \equiv 2, 10 \pmod{12} \\ k+2 & \text{if } k \equiv 0, 3, 4, 6, 8, 9 \pmod{12}. \end{cases}$$

Further, if $n \equiv 2, 4 \pmod{6}$, then $\lceil n/3 \rceil \leq \gamma(Q_n^t) \leq \frac{n}{2}$. However, if $n \equiv 1, 5 \pmod{6}$, then no bounds for $\gamma(Q_n^t)$ other than the trivial bounds $\lceil n/3 \rceil \leq \gamma(Q_n^t) \leq \gamma(Q_n)$ are known. We use the construction in Section 3 to obtain an upper bound for $\gamma(Q_n^t)$ not depending on $\gamma(Q_n)$ for certain values of n .

Note that on toroidal boards, all queens in D' lie on $Q_{n'}^t$ and all lines covered by infeasible queens on the ordinary board are now covered by feasible queens. Hence Lemmas 4–6 hold in this case too, while Lemma 8 changes to

Lemma 13. *D' has queens on every second row and column of $Q_{n'}^t$, except on the orthogonals through the centre of each copy.*

Hence, following the proof of Theorem 10, using D' instead of D^* , we construct the sets

$$Y = D' \cup M \cup R,$$

$$Y' = D' \cup M' \cup R',$$

which clearly dominate $Q_{n'}^t$. Moreover,

$$\begin{aligned} |Y| &= |D'| + |M| + |R| \\ &= (2k+1)(6m-2) + 2(6m-2) + \text{a constant} \\ &= 6m(2k+3) + \text{a constant}, \end{aligned}$$

similarly $|Y'| = 6m(2k+3) + \text{a constant}$. This gives $\gamma(Q_{n'}^t) \leq [(2k+3)/(4k+2)]n' + \mathcal{O}(1)$. Thus we have

Corollary 14. *If there exists a dominating (i, i) - or $(i, i+2)$ -set ($i < k$) of Q_{4k+1} of cardinality $2k+1$, then for any $m \geq 1$ and $n = 2(6m-1)(2k+1) - 1$, $\gamma(Q_n^t) \leq [(2k+3)/(4k+2)]n + \mathcal{O}(1)$.*

Corollary 14 gives an improved bound for $\gamma(Q_n^t)$ only when $n = 2(6m-1)(2k+1) - 1, 5 \pmod{6}$, that is, only when $k \equiv -1, 1 \pmod{3}$. Moreover, since for most values of i the s -diagonal $s=i$ (d -diagonal $d=i$) of Q_n^t is not contained in the s -diagonal $s=i$ (d -diagonal $d=i$) of Q_{n+1}^t , Corollary 14 does not provide an upper bound for $\gamma(Q_n^t)$ for arbitrary n large enough, unlike Corollary 11 does for $\gamma(Q_n)$.

Acknowledgements

This paper was written while A.P. Burger and C.M. Mynhardt were employed by the University of South Africa in the Department of Mathematics, Applied Mathematics and Astronomy. Research grants by the University and the South African National Research Foundation are gratefully acknowledged.

References

- [1] A.P. Burger, E.J. Cockayne, C.M. Mynhardt, Domination numbers for the queen's graph, *Bull. Inst. Combin. Appl.* 10 (1994) 73–82.
- [2] A.P. Burger, E.J. Cockayne, C.M. Mynhardt, Domination and irredundance in the queen's graph, *Discrete Math.* 163 (1997) 47–66.
- [3] A.P. Burger, E.J. Cockayne, C.M. Mynhardt, Queens graphs for chessboards on the torus, *Australas. J. Combin.* 24 (2001) 231–246.
- [4] A.P. Burger, C.M. Mynhardt, Symmetry and domination in queens graphs, *Bull. Inst. Combin. Appl.* 29 (2000) 11–24.
- [5] A.P. Burger, C.M. Mynhardt, Properties of dominating sets of the queens graph Q_{4k+3} , *Utilitas Math.* 57 (2000) 237–253.
- [6] A.P. Burger, C.M. Mynhardt, An upper bound for the minimum number of queens covering the $n \times n$ chessboard, *Discrete Appl. Math.* 121 (2002) 51–60.
- [7] A.P. Burger, C.M. Mynhardt, W.D. Weakley, Queens graphs for chessboards on the torus, *Australas. J. Combin.* 24 (2001) 231–246.
- [8] E.J. Cockayne, Chessboard domination problems, *Discrete Math.* 86 (1990) 13–20.
- [9] C.F. de Jaenisch, *Applications de l'Analyse Mathématique au Jeu des Echecs*. Petrograd, 1862.
- [10] P.B. Gibbons, J.A. Webb, Some new results for the queens domination problem, *Austral. J. Combin.* 15 (1997) 145–160.

- [11] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel-Dekker, New York, 1998.
- [12] M.D. Kears, P.B. Gibbons, Computational methods and new results for chessboard problems, *Austral. J. Combin.* 23 (2001) 253–284.
- [13] C.M. Mynhardt, Upper bounds for the domination number of toroidal Queens graphs, *Discussiones Math.* 23 (2003), to appear.
- [14] P.R.J. Östergård, W.D. Weakley, Values of domination numbers of the queen's graph, preprint.
- [15] W.D. Weakley, Domination in the queen's graph, in: Y. Alavi, A.J. Schwenk, (Eds.), *Graph Theory, Combinatorics, and Algorithms*, Vol. 2, Wiley-Interscience, New York, 1995, pp. 1223–1232.
- [16] W.D. Weakley, Upper bounds for domination numbers of the queen's graph, *Discrete Math.* 242 (2002) 229–243.
- [17] W.D. Weakley, A lower bound for domination numbers of the queen's graph, *J. Combin. Math. Combin. Comput.*, to appear.